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SHOCK MODELS AND THE MIFRA PROPERTY

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Marshall and Shaked [6] have shown that some multivariate life distributions obtained from their shock model satisfy the IFRA conditions A and B of Esary and Marshall [5]. Block and Savits [2] have introduced a multivariate IFRA condition which is stronger than Conditions A and B. In this paper it is shown that the multivariate life distributions of Marshall and Shaked actually satisfy the Block–Savits MIFRA condition. As a consequence it follows that the damage processes associated with the Marshall–Shaked shock models are multivariate strongly IFRA in the sense of Block and Savits [3].

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Multivariate IFRA distributions	coherent structures
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0. Introduction

Recently, Marshall and Shaked [6] studied the properties of a multivariate life distribution derived from a cumulative damage shock model. The model they considered is described below. Consider n devices which are subjected to shocks governed by a Poisson process. Upon occurrence of the ith shock, all devices suffer a non-negative random damage $\boldsymbol{Y}_i$ with joint distribution F_i. It is assumed that damages from successive shocks accumulate additively and are independent. A device fails when its accumulated damage exceeds its breaking threshold. Let T_i denote the lifetime of the ith device.

Marshall and Shaked [6] were able to show that if damages from successive shocks are stochastically increasing in an appropriate sense (e.g., if $\mathrm{E}[f(\boldsymbol{Y}_i)] \leq \mathrm{E}[f(\boldsymbol{Y}_{i+1})]$, $i = 1, 2, \ldots$, for all increasing real functions f), then $(T_1, \ldots, T_n)$ satisfies Condition B of Esary and Marshall [5] (see exact definition in Section 1). They also showed that in the case $F_1 = F_2 = \cdots$, the vector $(T_1, \ldots T_n)$ satisfies Condition A of Esary and Marshall [5] (see definition in Section 1).

In this paper, we will be concerned with multivariate distributions which are 'multivariate increasing failure rate average' (MIFRA) in the sense of Block and Savits [2] (see definition in Section 1). In Section 2, we show that actually $(T_1, \ldots, T_n)$ is MIFRA in the case $F_1 = F_2 = \cdots$. In [2] it is shown that MIFRA is stronger than Conditions A and B.

In Section 3 we consider a shock model for an IFRA process. Ross [7] first introduced the notion of a univariate IFRA process. This notion was generalized to n-dimensions recently by Block and Savits [3].

Section 1 contains some preliminaries.

As usual, we say 'increasing' for 'nondecreasing', and 'decreasing' for 'nonincreasing'.

1. Definitions

A function τ defined on $\{\boldsymbol{t} = (t_1, \ldots, t_n)\colon t_1 \geqslant 0, \ldots, t_n \geqslant 0\}$ is a *coherent life function* of order n if for some positive integer p and some sets $P_j \subset \{1, 2, \ldots, n\}$, $j = 1, \ldots, p$, τ has the representation

$$\tau(\boldsymbol{t}) = \max_{i \leqslant j \leqslant p} \min_{i \in P_j} t_i. \tag{1.1}$$

Equivalently, τ is a coherent life function if for some positive integer k and some sets $K_j \subset \{1, 2, \ldots, n\}$, $j = 1, \ldots, k$, τ has the representation

$$\tau(\boldsymbol{t}) = \min_{1 \leqslant j \leqslant k} \min_{1 \in K_j} t_i.$$

See Esary and Marshall [4] for a discussion about the role of coherent life functions in reliability theory.

Let τ have the representation (1.1). Its dual, τ^{D}, is defined by

$$\tau^{\mathrm{D}}(\boldsymbol{t}) = \min_{1 \leqslant j \leqslant p} \max_{1 \in P_j} t_i$$

and is also a coherent life function (see Marshall and Shaked [6]).

A non-negative random variable T (or its survival function $\bar{F}(t) = \mathbf{P}(T > t)$) is said to be IFRA or IHRA (increasing failure (or hazard) rate average) if

$$\bar{F}(\alpha t) \geqslant [\bar{F}(t)]^{\alpha} \quad \text{for all } 0 < \alpha \leqslant 1 \text{ and } t \geqslant 0.$$

If $T_1, \ldots, T_n$ are independent IFRA random variables and τ is a coherent life function of order n, then $\tau(\boldsymbol{T})$ is IFRA (see Barlow and Proschan [1, p. 104]). The class of IFRA distributions is the smallest class that satisfies this closure property (see [1, p. 89]). The importance of the IFRA class stems from this fact.

Esary and Marshall [5] introduced some analogs of the IFRA property in the multivariate setting. Two of them are the following.

1.2. Definition. The random vector $\boldsymbol{T}=(T_1,\ldots,T_n)$ with survival function $\bar{F}(\boldsymbol{t})=\mathbf{P}(T_1>t_1,\ldots,T_n>t_n)$ is said to satisfy *Condition A* if

$$\bar{F}(\alpha\boldsymbol{t})\geqslant[\bar{F}(\boldsymbol{t})]^{\alpha}\quad\text{for all } 0<\alpha\leqslant 1 \text{ and } \boldsymbol{t}\geqslant\boldsymbol{0}.$$

1.3. Definition. The random vector $\boldsymbol{T}$ is said to satisfy *Condition B* if for every coherent life function τ, $\tau(\boldsymbol{T})$ is IFRA.

The following stronger multivariate IFRA condition was introduced by Block and Savits [2].

1.4. Definition. The random vector $(T_1,\ldots,T_n)$ is said to be *MIFRA* if for every continuous, non-negative increasing function h and every $0<\alpha\leqslant 1$

$$\mathbf{E}[h(T_1,\ldots,T_n)]\leqslant\mathbf{E}^{1/\alpha}[h^{\alpha}(T_1/\alpha,\ldots,T_n/\alpha)].$$

It is already known that the shock model of Marshall and Shaked [6] yields lifelengths that satisfy Conditions A and B. In the next section it will be shown that these lifelengths actually satisfy the MIFRA property which is a strictly stronger condition.

2. Shock models and the MIFRA property

We first describe the shock model in more detail, essentially following the notation of Marshall and Shaked [6]. The process $\{N(t), t\geqslant 0\}$ which governs the arrival of shocks is assumed to be a Poisson process with intensity λ. At the occurrence of the ith shock, the jth device suffers a nonnegative random damage Y_{ij}. We denote by F_i the distribution of $\boldsymbol{Y}_i=(Y_{i1},\ldots,Y_{in})$. We assume that the $\boldsymbol{Y}_i$'s are independent and are also independent of $\{N(t), t\geqslant 0\}$. The accumulated damage on the jth device as a result of the first k shocks is denoted by $S_j^{(k)}$. Thus $S_j^{(k)}=\sum_{i=1}^{k}Y_{ij}$. The jth device fails when the accumulated damage on the jth device exceeds its breaking threshold y_j. Thus, if T_j is the lifetime of the jth device, then

$$T_j=\inf\{t\geqslant 0\colon S_j^{(N(t))}>y_j\}. \tag{2.1}$$

We let $S_j^{(0)}\equiv 0$, $1\leqslant j\leqslant n$. Thus $y_j=0$ means that the jth device fails upon the occurrence of any positive damage. The joint survival of $T_1,\ldots,T_n$ is determined by the constants $\lambda, y_1, y_2,\ldots,y_n$ and the distributions $F_1, F_2,\ldots$; it will be denoted by

$$\mathbf{P}(T_1>t_1,\ldots,T_n>t_n)=\bar{H}_n(\boldsymbol{t};\lambda,y_1,\ldots,y_n;F_1,F_2,\ldots).$$

An explicit expression for $\bar{H}_n$ is given in [6].

The next theorem is the main result of this section.

2.2. Theorem. *Assume that the joint survival of* $(T_1, \ldots, T_n)$ *is* $\bar{H}_n(\cdot\,; \lambda, y_1, \ldots, y_n; F_1, F_2, \ldots)$. *If* $F_1 = F_2 = \cdots = F$ *for some* F, *then* $(T_1, \ldots, T_n)$ *is MIFRA.*

The following lemmas will be used in the proof of the theorem.

Lemma 2.1 of [6] generalizes to give:

2.3. Lemma. *Assume that the joint survival of* $(T_1, \ldots, T_n)$ *is* $\bar{H}_n(\cdot\,; \lambda, y_1, \ldots, y_n; F_1, F_2, \ldots)$. *Let* τ *be a coherent life function of order* n *and let* $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ *be constants. Then for* $t > 0$,

$$\mathbf{P}\{\tau(a_1T_1, \ldots, a_nT_n) > t\} =$$

$$= \sum_{k_1=0}^{\infty} \sum_{k_2=k_1}^{\infty} \cdots \sum_{k_n=k_{n-1}}^{\infty} \prod_{j=1}^{n} \mathrm{e}^{-\lambda(t_j - t_{j-1})} \cdot \frac{(\lambda(t_j - t_{j-1}))^{k_j - k_{j-1}}}{(k_j - k_{j-1})!}$$

$$\times \mathbf{P}\left\{\tau^{\mathrm{D}}\left(\frac{S_1^{(k_1)}}{y_1}, \ldots, \frac{S_n^{(k_n)}}{y_n}\right) \leq 1\right\} \tag{2.4}$$

where on the right-hand side of (2.4) *we define* $0/0 \equiv 1$, $k_0 = 0$, $t_0 = 0$ *and* $t_j = t/a_j$ *for* $1 \leq j \leq n$, *and where* τ^{D} *denotes the dual life function of* τ (*see Section* 1).

2.5. Remark. If for some $n_1 \leq n$, $a_{n_1} > a_{n_1+1} = \cdots = a_n = 0$ in Lemma 2.3, then (2.4) holds with n_1 replacing n.

Let C be a subset of $\mathbb{R}^n$ such that

$$\boldsymbol{u}, \boldsymbol{v} \in C \Rightarrow \boldsymbol{u} + \boldsymbol{v} \in C. \tag{2.6}$$

Denote by $\mathcal{B}$ the set of all Borel subsets A of $\mathbb{R}^n$ that satisfy

$$\boldsymbol{u} \in A, \boldsymbol{z} \in C \Rightarrow \boldsymbol{u} - \boldsymbol{z} \in A. \tag{2.7}$$

For purposes of this paper it is sufficient to take $C = [0, \infty)^n$. Then $\mathcal{B}$ is the set of 'decreasing sets', and in particular all complements of upper domains (which are open 'upper sets' (see Section 3)) are in $\mathcal{B}$.

Let $0 \leq \theta_j \leq 1$, $1 \leq j \leq n$, be constants such that $\sum_{j=1}^n \theta_j = 1$. For every Borel set A in $\mathbb{R}^n$ define

$$\tilde{P}_k(A) = \sum_{l_1=0}^{k} \sum_{l_2=l_1}^{k} \cdots \sum_{l_{n-1}=l_{n-2}}^{k} \binom{k}{l_1, l_2 - l_1, \ldots, l_n - l_{n-1}} \prod_{j=1}^{n} \theta_j^{l_j - l_{j-1}}$$

$$\times \mathbf{P}\{(S_1^{(l_1)}, S_2^{(l_2)}, \ldots, S_n^{(l_n)}) \in A\}. \tag{2.8}$$

Here $l_0 = 0$ and $l_n = k$. Note that in the following lemma it is assumed, as in Theorem 2.2, that the $\boldsymbol{Y}_i$'s are identically distributed.

2.9. Lemma. *Assume that the joint survival of* $(T_1, \ldots, T_n)$ *is* $\bar{H}_n(\cdot\,; \lambda, y_1, \ldots, y_n; F, F, \ldots)$ *for some* F *and* $\lambda > 0$. *If*

$$\mathbf{P}\{\boldsymbol{Y}_i \in C\} = 1, \tag{2.10}$$

then for every $A \in \mathscr{B}$

$$[\bar{P}_k(A)]^{1/k} \text{ is decreasing in } k = 1, 2, \ldots. \tag{2.11}$$

Let m be an integer. In the proof of Theorem 2.2 use will be made of the observation that if $T_1, \ldots, T_n$ can be represented as lifetimes of devices subjected to a shock model as described above, then also

$$\underbrace{T_1, \ldots, T_1}_{m \text{ times}}, \underbrace{T_2, \ldots, T_2}_{m \text{ times}}, \ldots, \underbrace{T_n, \ldots, T_n}_{m \text{ times}}$$

can be such represented. The next lemma is a formal statement of a stronger result that will be used in Section 3.

2.12. Lemma. *Let* $N(t)$ *be a Poisson process of intensity* λ. *Let* y_{jl}, $1 \leqslant j \leqslant n$, $1 \leqslant l \leqslant m$ *be nonnegative numbers. Let* $\mathbf{Y}_i = (Y_{i1}, \ldots, Y_{in})$ *be a nonnegative random vector distributed according to* F_i, $i = 1, 2, \ldots$, *and assume that the* $\mathbf{Y}_i$*'s are independent and are also independent of* $N(t)$. *Denote* $S_j^{(k)} = \sum_{i=1}^{k} Y_{ij}$ *and set*

$$\tilde{T}_{jl} = \inf\{t \geqslant 0 \colon S_j^{(N(t))} > y_{jl}\}.$$

Then

$$\mathbf{P}\{\tilde{T}_{jl} > t_{jl}, 1 \leqslant j \leqslant n, 1 \leqslant l \leqslant m\} =$$

$$= \bar{H}_{nm}(t_{11}, \ldots, t_{jl}, \ldots, t_{nm}; \lambda, y_{11}, \ldots, y_{jl}, \ldots, y_{nm}; \tilde{F}_1, \tilde{F}_2, \ldots) \tag{2.13}$$

where $\tilde{F}_i$ *is an mn-dimensional distribution defined by*

$$\tilde{F}_i(z_{11}, \ldots, z_{jl}, \ldots, z_{nm}) = F_i(\min_{1 \leqslant l \leqslant m} (z_{1l}), \ldots, \min_{1 \leqslant l \leqslant m} (z_{nl})). \tag{2.14}$$

2.15. Remark. Taking $y_{j1} = \cdots = y_{jm} = y_j$, $1 \leqslant j \leqslant n$, in Lemma 2.12, it follows that the survival of

$$(\underbrace{T_1, \ldots, T_1}_{m \text{ times}}, \ldots, \underbrace{T_n, \ldots, T_n}_{m \text{ times}})$$

is of the form (2.13), where $T_1, \ldots, T_n$ are defined in (2.1).

Having these lemmas we are now ready to prove Theorem 2.2. The proofs of the lemmas will be given at the end of the section.

Proof of Theorem 2.2. First it will be shown that for every choice of constants $a_j \geqslant 0$, $1 \leqslant j \leqslant n$, and for every coherent life function τ of order n

$$T \equiv \tau(a_1 T_1, \ldots, a_n T_n) \text{ is IFRA}. \tag{2.16}$$

Without loss of generality we can assume that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 0$ since otherwise the a_j's can be such ordered by renumbering the T_j's. Also we may assume that

$a_n > 0$, because if some of the a_i's are zero, then the problem is reduced to an identical one in a lower dimension.

Let $A = \{z : \tau^{\mathrm{D}}(z_1/y_1, \ldots, z_n/y_n) \leq 1\}$. From (2.4) it is easy to see that

$$\mathbf{P}(T > t) = \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda t_n} \frac{(\lambda t_n)^k}{k!} \bar{P}_k(A), \tag{2.17}$$

where $\bar{P}_k(A)$ is given in (2.8) with $\theta_i = (a_i^{-1} - a_{i-1}^{-1})/a_n^{-1}$, $i = 1, 2, \ldots, n$ (here $a_0^{-1} \equiv 0$) and $t_n = t/a_n$.

As is remarked above, taking $C = [0, \infty)^n$, it follows that $A \in \mathscr{B}$. Thus, Lemma 2.9 applies. From the representation (2.17) and property (2.11) it follows (using, e.g., Lemma 1.2 of [6]) that T is IFRA.

The preceding argument shows that whenever the survival of $(T_1, \ldots, T_n)$ is of the form $\bar{H}_n(\cdot\,; \lambda, y_1, \ldots, y_n; F, F, \ldots)$, then (2.16) holds. By Lemma 2.12, the survival of the random vector

$$(\underbrace{T_1, \ldots, T_1}_{m \text{ times}}, \ldots, \underbrace{T_n, \ldots, T_n}_{m \text{ times}})$$

is of this form (that $\tilde{F}_1 = \tilde{F}_2 = \cdots$ follows from (2.14) and the assumption $F_1 = F_2 = \cdots$). Hence, by (2.16), for every coherent life function τ of order mn and non-negative numbers a_{jl}, $1 \leq j \leq n$, $1 \leq l \leq m$,

$$\tau(a_{11}T_1, \ldots, a_{jl}T_j, \ldots, a_{nm}T_n) \text{ is IFRA.} \tag{2.18}$$

It is shown in Block and Savits [2] that (2.18) is equivalent to $(T_1, \ldots, T_n)$ being MIFRA.

The proofs of Lemmas 2.9 and 2.12 will now be given. The proof of Lemma 2.3 can be obtained by modifying the proof of Lemma 3.1 in [6]; we omit the details.

Proof of Lemma 2.9. As in [6] it can be shown, using (2.7) and (2.10), that

$$\mathbf{P}\{\mathbf{Y}_i \in A - \boldsymbol{v}\} = 0 \quad \text{for all } \boldsymbol{v} \notin A, A \in \mathscr{B}.$$

Similarly, by virtue of (2.6), for integers $l_1, \ldots, l_n$,

$$\mathbf{P}\{(S_1^{(l_1)}, \ldots, S_n^{(l_n)}) \in A - \boldsymbol{v}\} = 0 \quad \text{for all } \boldsymbol{v} \notin A, A \in \mathscr{B}. \tag{2.19}$$

Furthermore if $\boldsymbol{v}_2 - \boldsymbol{v}_1 \in C$ and $\boldsymbol{u} \in A$, then, by (2.7), $\boldsymbol{u} + \boldsymbol{v}_1 = \boldsymbol{u} - (\boldsymbol{v}_2 - \boldsymbol{v}_1) + \boldsymbol{v}_2 \in A + \boldsymbol{v}_2$, that is,

$$A + \boldsymbol{v}_1 \subset A + \boldsymbol{v}_2 \quad \text{whenever } \boldsymbol{v}_2 - \boldsymbol{v}_1 \in C, A \in \mathscr{B}. \tag{2.20}$$

In particular, since $A \in \mathscr{B} \Rightarrow A + \boldsymbol{v} \in \mathscr{B}$ for every $\boldsymbol{v} \in \mathbb{R}^n$, it follows from (2.20) that

$$A - \boldsymbol{v} \subset A \quad \text{whenever } \boldsymbol{v} \in C, A \in \mathscr{B}. \tag{2.21}$$

It is easy to see now from (2.8), using (2.21), that

$$\bar{P}_k(A - \boldsymbol{v}) \leq \bar{P}_k(A) \quad \text{for every } \boldsymbol{v} \in C, A \in \mathscr{B}. \tag{2.22}$$

For purposes of an inductive argument in proving (2.11) it is useful to express $\bar{P}_{k+1}$ in terms of $\bar{P}_k$. Using a well-known combinatorial identity write, with $l_0=0$ and $l_n=k+1$,

$$\bar{P}_{k+1}(A)=\sum_{l_1=0}^{k+1}\sum_{l_2=l_1}^{k+1}\cdots\sum_{l_{n-1}=l_{n-2}}^{k+1}\left[\binom{k}{l_1-1,\,l_2-l_1,\ldots,k+1-l_{n-1}}\right.$$
$$\left.+\binom{k}{l_1,\,l_2-l_1-1,\,l_3-l_2,\ldots,k+1-l_{n-1}}+\cdots+\binom{k}{l_1,\,l_2-l_1,\ldots,k-l_{n-1}}\right]$$
$$\times\prod_{j=1}^{n}\theta_j^{l_j-l_{j-1}}\mathbf{P}\{(S_1^{(l_1)},\ldots,S_n^{(l_n)})\in A\},$$

where

$$\binom{k}{m_1,m_2,\ldots,m_n}=0$$

whenever $\sum_{j=1}^{n}m_j=k$ and $m_j=-1$ for some $j\in\{1,\ldots,n\}$. Split the last sum into n sums in an obvious way. Substitute $l_1'=l_1-1$, $l_2'=l_2-1,\ldots,l_{n-1}'=l_{n-1}-1$ and $l_n'=l_n-1(=k)$ in the first sum. Substitute $l_2'=l_2-1,\ldots,l_{n-1}=l_{n-1}'-1$ and $l_n'=l_n-1(=k)$ in the second sum. Continuing in this way, substitute finally $l_n'=l_n-1(=k)$ in the nth sum. Leaving out the primes, we thus obtain the expression

$$\bar{P}_{k+1}(A)=\sum_{l_1=0}^{k}\sum_{l_2=l_1}^{k}\cdots\sum_{l_{n-1}=l_{n-2}}^{k}\binom{k}{l_1,\,l_2-l_1,\ldots,l_n-l_{n-1}}\prod_{i=1}^{n}\theta_i^{l_i-l_{i-1}}$$
$$\times\sum_{j=1}^{n}\theta_j\mathbf{P}\{(S_1^{(l_1)},\ldots,S_{j-1}^{(l_{j-1})},S_j^{(l_j+1)},\ldots,S_n^{(l_n+1)})\in A\}\tag{2.23}$$

with $l_0\equiv 0$ and $l_n\equiv k$.

Now, for $1\leqslant j\leqslant n$,

$$\mathbf{P}\{(S_1^{(l_1)},\ldots,S_{j-1}^{(l_{j-1})},S_j^{(l_j+1)},\ldots,S_n^{(l_n+1)})\in A\}=$$
$$=\int_C\cdots\int\mathbf{P}\{(S_1^{(l_1)},\ldots,S_n^{(l_n)})\in A-(0,\ldots,0,v_j,v_{j+1},\ldots,v_n)\}\,\mathrm{d}F(v)$$
$$\underset{(2.19)}{=}\int_{\{v\in C:\,(0,\ldots,0,v_j,\ldots,v_n)\in A\}}\cdots\int\mathbf{P}\{(S_1^{(l_1)},\ldots,S_n^{(l_n)})\in A$$
$$-(0,\ldots,0,v_j,v_{j+1},\ldots,v_n)\}\,\mathrm{d}F(v).$$

Combining this with (2.23) yields

$$\bar{P}_{k+1}(A)=\sum_{j=1}^{n}\theta_j\int_{\{v\in C:(0,\ldots,0,v_j,\ldots,v_n)\in A\}}\cdots\int\bar{P}_k[A-(0,\ldots,0,v_j,v_{j+1},\ldots,v_n)]\,\mathrm{d}F(v).\tag{2.24}$$

Finally, proceeding to the proof of (2.11) note that by definition

$$\bar{P}_1(A)=\sum_{j=1}^{n}\theta_j\mathbf{P}\{(S_1^{(0)},\ldots,S_{j-1}^{(0)},S_j^{(1)},\ldots,S_n^{(1)})\in A\}. \tag{2.25}$$

Thus

$$\bar{P}_2(A)\underset{(2.24)}{=}\sum_{j=1}^{n}\theta_j\int\cdots\int_{\{v\in C:\,(0,\ldots,0,v_j,\ldots,v_n)\in A\}}\bar{P}_1[A-(0,\ldots,0,v_j,v_{j+1},\ldots,v_n)]\,\mathrm{d}F(v)$$

$$\underset{(2.22)}{\leqslant}\bar{P}_1(A)\sum_{j=1}^{n}\theta_j\,\mathbf{P}\{(S_1^{(0)},\ldots,S_{j-1}^{(0)},S_j^{(1)},\ldots,S_n^{(1)})\in A\}$$

$$\underset{(2.25)}{=}[\bar{P}_1(A)]^2.$$

Now assume that

$$[\bar{P}_{k-1}(A)]^{1/(k-1)}\geqslant[\bar{P}_k(A)]^{1/k}\quad\text{for all } A\in\mathscr{B}. \tag{2.26}$$

Then

$$[\bar{P}_k(A)]^{k+1}\underset{(2.24)}{=}\bar{P}_k(A)\left\{\sum_{j=1}^{n}\theta_j\int\cdots\int_{\{v\in C:\,(0,\ldots,0,v_j,\ldots,v_n)\in A\}}\bar{P}_{k-1}[A-(0,\ldots,0,v_j,v_{j+1},\ldots,v_n)]\,\mathrm{d}F(v)\right\}^k$$

$$\underset{(2.26)}{\geqslant}\left\{\sum_{j=1}^{n}\theta_j\int\cdots\int_{\{v\in C:\,(0,\ldots,0,v_j,\ldots,v_n)\in A\}}[\bar{P}_k(A)]^{1/k}\times\bar{P}_k[A-(0,\ldots,0,v_j,v_{j+1},\ldots,v_n)]^{(k-1)/k}\,\mathrm{d}F(v)\right\}^k$$

$$\underset{(2.22)}{\geqslant}\left\{\sum_{j=1}^{n}\theta_j\int\cdots\int_{\{v\in C:\,(0,\ldots,0,v_j,\ldots,v_n)\in A\}}\bar{P}_k[A-(0,\ldots,0,v_j,v_{j+1},\ldots,v_n)]\,\mathrm{d}F(v)\right\}^k$$

$$\underset{(2.24)}{=}[\bar{P}_{k+1}(A)]^k.$$

Proof of Lemma 2.12. Eq. (2.13) is easily obtained from the observation that $\tilde{F}_i$, which is defined in (2.14), is the distribution function of the mn-dimensional random vector

$$(\underbrace{Y_{i1},\ldots,Y_{i1}}_{m\text{ times}},\ldots,\underbrace{Y_{in},\ldots,Y_{in}}_{m\text{ times}}).$$

3. IFRA processes from shock models

Let $Z(t)$ be a right-continuous nonnegative increasing process. According to Ross [7], $Z(t)$ is called an IFRA process if the random variable $T_a=\inf\{t\geqslant 0: Z(t)>a\}$ is

IFRA for every $a \geq 0$. In [3], Block and Savits generalized this notion to the n-dimensional case as follows (see Remark 3.5 at the end of this section). Let $Z_j(t)$, $1 \leq j \leq n$, be right-continuous nonnegative increasing processes, and set $\boldsymbol{Z}(t) = (Z_1(t), \ldots, Z_n(t))$. An upper domain $U \subset \mathbb{R}^n$ is an open set with the property that $\boldsymbol{y} \geq \boldsymbol{x} \in U \Rightarrow \boldsymbol{y} \in U$. We say that $\boldsymbol{Z}(t)$ is an IFRA process if and only if the random variable $T_U = \inf\{t \geq 0: \boldsymbol{Z}(t) \in U\}$ is IFRA for all upper domains U. Let $B_j \in \mathbb{R}$ be the state space of $Z_j(t)$, $1 \leq j \leq n$. Then according to [3], $\boldsymbol{Z}(t)$ is an IFRA process if and only if for every choice of an integer m and numbers $y_{jl} \in B_j$, $1 \leq j \leq n$, $1 \leq l \leq m$, the random variables $\{T_{jl}; 1 \leq j \leq n, 1 \leq l \leq m\}$ satisfy Condition B of [5], where $T_{jl} = \inf\{t \geq 0: Z_j(t) > y_{jl}\}$. In the case that the collection $\{T_{jl}; 1 \leq j \leq n, 1 \leq l \leq m\}$ is MIFRA we say that the process is strongly IFRA.

3.1. Remark. If $(Z_1(t), \ldots, Z_n(t))$ is an IFRA (a strongly IFRA) process, then for every choice of increasing nonnegative functions ϕ_j on B_j such that $\phi_j(Z_j(t))$ is right-continuous, $1 \leq j \leq n$, the process $(\phi_1(Z_1(t)), \ldots, \phi_n(Z_n(t)))$ is an IFRA (a strongly IFRA) process.

Consider the shock model of Section 2. The damage process associated with this model is $\{(S_1^{(N(t))}, \ldots, S_n^{(N(t))}), t \geq 0\}$. As usual, if F_i and F_{i+1} are n-dimensional distributions, we say that F_{i+1} is stochastically larger than F_i if

$$\int_{\mathbb{R}^n} \cdots \int f(\boldsymbol{y})\, \mathrm{d}F_{i+1}(\boldsymbol{y}) \geq \int_{\mathbb{R}^n} \cdots \int f(\boldsymbol{y})\, \mathrm{d}F_i(\boldsymbol{y})$$

for all measurable increasing real functions f. Notation $F_{i+1} \overset{\text{st}}{\geq} F_i$.

3.2. Theorem. *If* $F_i \overset{\text{st}}{\leq} F_{i+1}$, $i = 1, 2, \ldots$, *then* $\{(S_1^{(N(t))}, \ldots, S_n^{(N(t))}), t \geq 0\}$ *is an IFRA process.*

Proof. Let m be an integer and let $y_{jl} \geq 0$, $1 \leq j \leq n$, $1 \leq l \leq m$ be real numbers. Define

$$T_{jl} = \inf\{t \geq 0: S_j^{(N(t))} > y_{jl}\}.$$

As mentioned earlier we only need to prove that $\{T_{jl}; 1 \leq j \leq n, 1 \leq l \leq m\}$ satisfy Condition B. By Lemma 2.12, $\{T_{jl}\}$ satisfy the conditions of Theorem 3.3 in [6], and this implies that they satisfy Condition B.

3.3. Theorem. *If* $F_1 = F_2 = \cdots$, *then* $\{(S_1^{(N(t))}, \ldots, S_n^{(N(t))}), t \geq 0\}$ *is a strongly IFRA process.*

Proof. Define T_{jl} as in the previous proof. By Lemma 2.12, $\{T_{jl}\}$ satisfy the conditions of Theorem 2.2, and this implies that they are MIFRA.

In many applications the performance of a device at time t may depend mainly on the damage that has been accumulated on this device by time t. The performance may usually be taken as a monotone function of the accumulated damage. Then, by Remark 3.1, the performance level process is also an IFRA process.

For example assume that the state space of the performance level of each device is discrete: $\{0, 1, 2, \ldots, m\}$, say. This can be the case if a new device is in level 0 (perfect functioning) and as the damage accumulates, the device deteriorates: the performance of the jth device is no better than at level l if the accumulated damage on it exceeds the threshold y_{jl}. Of course $0 \leqslant y_{jl} \leqslant y_{j2} \leqslant \cdots \leqslant y_{jm}$.

Define the performance process $(X_1(t), \ldots, X_n(t))$ by

$$X_j(t) = \begin{cases} 0 & \text{if } S_j^{(N(t))} \leqslant y_{j1}, \\ l & \text{if } y_{jl} < S_j^{(N(t))} \leqslant y_{j,l+1},\ 1 \leqslant l \leqslant m \end{cases}$$

where $y_{j,m+1} \equiv \infty$.

3.4. Corollary. (a) *If* $F_i \overset{\text{st}}{\leqslant} F_{i+1}$, $i = 1, 2, \ldots$, *then* $\boldsymbol{X}(t)$ *is an IFRA process*;
(b) *if* $F_1 = F_2 = \cdots$, *then* $\boldsymbol{X}(t)$ *is a strongly IFRA process.*

Proof. To prove (a) note that for each $1 \leqslant j \leqslant n$, $X_j(t)$ is an increasing function of $S_j^{(N(t))}$. The result now follows from Remark 3.1 and Theorem 3.2. The proof of (b) is similar using Remark 3.1 and Theorem 3.3.

3.5. Remark. It should be mentioned that, in [3], IFRA processes were assumed to be decreasing processes instead of increasing processes, and the corresponding random variables T_U, for U an upper domain, were defined by $T_U = \inf\{t \geqslant 0: \boldsymbol{Z}(t) \notin U\}$. For our purposes it seems more intuitive to use increasing processes. Thus, in referring to [3], it is necessary to make appropriate modifications. Also, because of a slight technicality, if one uses increasing processes instead of decreasing processes, the random variables $T_U = \inf\{t \geqslant 0: \boldsymbol{Z}(t) \in U\}$ should be defined using closed upper sets instead of open upper sets. However, it is not hard to show that the random variables $\{T_U: U \text{ closed upper set}\}$ are IFRA if and only if $\{T_U: U \text{ open upper set}\}$ are IFRA.

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